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ON VIBRATIONS OF A PLATE

MOVING IN A GAS

By A. A. Movchan

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ON VIBRATIONS OF A PLATE MOVING IN A GAS *

A. A. Movchan

The vibrations are investigated of a cantilever plate of infinite width moving in a gas with large supersonic velocity in the direction from the clamped to the free edge. The pressure of the gas is taken into account by an approximate formula.

Under the conditions of the two-dimensional problem the relation is studied between the characteristics of the natural vibrations and the velocity of motion of the plate and certain conclusions drawn on the stability of the motion. A formula for the critical velocity is obtained.

The formulation of the problem here considered is due to A. A. Ilyushin to whom the author wishes to express his deep-felt thanks for his help and valuable remarks.

1. Fundamental equations. The acting forces and the geometrical constraints on the boundary of the plate here studied permit it to move in its plane rectilinearly with constant velocity c in a certain gas medium. In the plane of this undisturbed motion we introduce a rectangular system of coordinates x, y moving rectilinearly together with the plate with velocity c along the x-axis (Fig. 1). The plate, clamped at the edge $x = 0$ and free at the edge $x = a$, is a strip of infinite width along the y-axis between these edges.

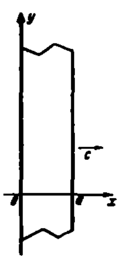

Fig. 1

The small deflection $w(x, y, t)$ of the points of the plate satisfies the equation

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = q \qquad \left(D = \frac{2Eh^3}{3(1-\nu^2)}\right)$$

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and the corresponding boundary conditions (ref. [1]. The transverse load $\underline{q}$ acting on the moving plate is made up of the inertia forces

$$q_1 = -2h\rho \frac{\partial^2 w}{\partial t^2}$$

and the aerodynamic forces[1]

$$q_2 = 2B\left(c\frac{\partial w}{\partial x} - \frac{\partial w}{\partial t}\right) \qquad \left(B = \frac{p_0 \varkappa}{c_0}\right)$$

where 2h is the thickness of the plate, ρ the density of the material of the plate, D the rigidity of the plate, E Young's modulus of the material of the plate, ν the Poisson ratio of the material of the plate, $\underline{c}$ the velocity of the undisturbed motion of the plate, B a nonnegative number characterizing the properties of the gas in which the plate moves, p_0 the pressure of the gas at infinity, κ the polytropic exponent of the gas, c_0 the velocity of sound in the gas at infinity.

We shall assume all quantities entering the problem to be independent of the coordinate $\underline{y}$ (two-dimensional problem). Then, introducing the quantity x/a and keeping for it the same notation $\underline{x}$ as before, the problem under consideration can be described by the equations

$$\frac{\partial^4 w}{\partial x^4} - \frac{2Ba^3c}{D}\frac{\partial w}{\partial x} + \frac{a^4}{D}\left(2B\frac{\partial w}{\partial t} + 2h\rho\frac{\partial^2 w}{\partial t^2}\right) = 0 \tag{1.1}$$
$$w = \frac{\partial w}{\partial x} = 0 \quad \text{for} \quad x = 0, \qquad \frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0 \quad \text{for} \quad x = 1$$

We shall consider solutions of the problem that are representable in the form

$$w(x, t) = X(x)e^{\omega t}, \qquad X(x) = |X(x)|e^{i\varphi(x)} \tag{1.2}$$

where X(x) is a complex function of a real variable $\underline{x}$, and $\omega = p + iq$ is a certain complex number which we shall call the complex frequency.

---

[1] This formula for the aerodynamic forces acting on a thin plate moving in a gas with large supersonic velocity was communicated to the author by A. A. Ilyushin.

The complex solution (1.2) breaks down into the real solutions

$$
\begin{aligned}
w(x, t) &= |X(x)|\, e^{pt} \cos[\varphi(x) + qt] \\
w(x, t) &= |X(x)|\, e^{pt} \sin[\varphi(x) + qt]
\end{aligned}
\tag{1.3}
$$

The motions of the plate corresponding to the real solutions (1.3) we shall term the natural motions of the plate.

Depending on $\underline{p}$ the amplitude of the natural motions $|X(x)|e^{pt}$ can decrease with time, remain unchanged, or increase. If the real frequency $\underline{q}$ is not equal to zero, the natural motions have the character of vibrations. These vibrations for ϕ = const. resemble standing waves while for $\phi \neq$ const. they resemble running waves the velocity of propagation of which $-q/(d\phi/dx)$ for $d\phi/dx \neq$ const. is different at different points of the plate.

Substituting (1.2) in (1.1) and introducing the notations

$$A = \frac{2Ba^3c}{D} = \frac{2a^3 p_0 \varkappa c}{Dc_0} \tag{1.4}$$

$$\lambda = -\frac{a^4}{D}(2B\omega + 2h\rho\omega^2) \tag{1.5}$$

we find that the function (1.2) is a solution of the problem if and only if X(x) is an eigenfunction of the boundary problem

$$
\begin{gathered}
\frac{d^4X}{dx^4} - A\frac{dX}{dx} = \lambda X \\
X = \frac{dX}{dx} = 0 \quad \text{for} \quad x = 1, \qquad \frac{d^2X}{dx^2} = \frac{d^3X}{dx^3} = 0 \quad \text{for} \quad x = 1
\end{gathered}
\tag{1.6}
$$

and the complex frequency ω is determined corresponding to the eigenvalue λ from relation (1.5):

$$\omega = -\frac{B}{2h\rho} \pm \frac{1}{2h\rho}\sqrt{B^2 - \frac{2h\rho D\lambda}{a^4}} \tag{1.7}$$

We shall call the parameter A in equation (1.6) the reduced velocity of motion of the plate or simply the velocity A.

The fundamental boundary problem (1.6) for $A \neq 0$ is not selfadjoint; hence its eigenvalues λ, generally speaking, can be situated in the complex plane and not only on the real axis.

For real eigenvalues λ the eigenfunctions X(x) of the fundamental boundary problem can be chosen to be real. Then in formulas (1.3) $\phi(x)$ = const. and the natural motions, if of a vibrational character, resemble standing waves.

For complex eigenvalues λ the corresponding functions X(x) are always complex so that in formulas (1.3) $\phi(x) \neq$ const. and the natural motions resemble running waves.

For any λ one of the values (1.7) of the complex frequency ω has a negative real part. The second value (1.7) has a negative real part if λ is located in the complex plane within a second degree parabola

$$\mathrm{Re}\,\lambda = \frac{h\rho D}{2a^4 B^2}(\mathrm{Im}\,\lambda)^2 \tag{1.8}$$

(Fig. 2), a zero real part if λ is located on the same parabola (1.8), and a positive real part if λ is located outside the parabola (1.8).

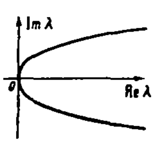

Fig. 2

The number of eigenvalues λ of the boundary problem that are located outside the limits of parabola (1.8) we shall denote as the degree of instability of the undisturbed motion, and the parabola (1.8) we shall denote as the stability parabola.

The eigenvalues λ of the fundamental boundary problem are connected with the corresponding eigenfunctions by the relation

$$\lambda = \left(\int_0^1 dx \frac{d^2X}{dx^2}\frac{d^2\overline{X}}{dx^2} - A\int_0^1 dx \frac{dX}{dx}\overline{X}\right) : \left(\int_0^1 dx\, X\overline{X}\right)$$

obtained by multiplying the first of equations (1.6) by $\overline{X}$ and integrating by parts, making use of the boundary conditions. From this relation it is seen that for A = 0 all eigenvalues λ of the fundamental boundary problem are real and positive.

2. <u>Change of eigenvalues of fundamental boundary problem with change of</u>

velocity of motion. For fixed A, λ any solution of equation (1.6) is a linear combination

$$X=\sum_{i=1}^{4} C_i X_i(A, \lambda; x) \tag{2.1}$$

of four of its linearly independent solutions $X_i(A, \lambda; x)$. We shall seek them in the form $X = e^{-zx}$, where $z = z(A, \lambda)$ are the roots of the characteristic equation

$$z^4 + Az - \lambda = 0 \tag{2.2}$$

Setting up expression (2.1) and subjecting it to the boundary conditions (1.6) it can easily be shown that the roots z_1, z_2, z_3, z_4 satisfy the equation

$$F(z_1, z_2, z_3, z_4) = \frac{\Delta}{\delta} = 0 \tag{2.3}$$

where

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 \\ z_1^2 e^{-z_1} & z_2^2 e^{-z_2} & z_3^2 e^{-z_3} & z_4^2 e^{-z_4} \\ z_1^3 e^{-z_1} & z_2^3 e^{-z_2} & z_3^3 e^{-z_3} & z_4^3 e^{-z_4} \end{vmatrix}$$

$$\delta = (z_1 - z_2)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_3 - z_4)$$

if and only if in equation (2.2) there is substituted for λ the eigenvalue of the fundamental boundary problem, for given A. The denominator in (2.3) is put in in order that the above mentioned property of the function F is maintained in the case of multiple roots of equation (2.2)

If in (2.3) there are substituted the explicit expressions for the roots $z_i = z_i(A, \lambda)$ as functions of A, λ there is obtained the equation

$$F(A, \lambda) = 0 \tag{2.4}$$

connecting the velocity A and the eigenvalues λ of the fundamental boundary problem. Equation (2.4) is very cumbersome. In order to simplify the investigation we shall in equation (2.4) pass from the parameters A, λ to other parameters. As the fundamental parameters we shall at first consider any two roots z_1, z_2 of equation (2.2). In terms of z_1, z_2 the roots z_3, z_4 and the parameters A, λ have the expression

$$z_{3,4} = -\frac{1}{2}(z_1 + z_2) \pm \sqrt{z_1 z_2 - \frac{3}{4}(z_1 + z_2)^2}$$
$$\lambda = -z_1 z_2 (z_1^2 + z_2^2 + z_1 z_2), \qquad A = -(z_1 + z_2)(z_1^2 + z_2^2)$$

Then from the parameters z_1, z_2 by the transformation

$$z_1 = \alpha + i\beta, \qquad z_2 = \alpha - i\beta \tag{2.5}$$

we pass to the parameters α, β.

$$z_{3,4} = -\alpha \pm \sqrt{\beta^2 - 2\alpha^2} \tag{2.6}$$

$$\lambda = (\alpha^2 + \beta^2)(\beta^2 - 3\alpha^2) \tag{2.7}$$

$$A = 4\alpha(\beta^2 - \alpha^2) \tag{2.8}$$

Finally, substituting in (2.3) the expressions (2.5) and (2.6) we obtain equation (2.4) expressed in terms of the parameters α, β:

$$F(\alpha, \beta) = \\ = \frac{\cos\beta\left[((\beta^2 - \alpha^2)^2 + 12\alpha^4)\,\mathrm{ch}\sqrt{\beta^2 - 2\alpha^2} + 2\alpha(\beta^4 - \alpha^4)\dfrac{\mathrm{sh}\sqrt{\beta^2 - 2\alpha^2}}{\sqrt{\beta^2 - 2\alpha^2}}\right] + \frac{1}{2}(\beta^2 + \alpha^2)^2 e^{-2\alpha}}{2\left[(\beta^2 - 3\alpha^2)^2 + 4\alpha^2\beta^2\right]} + \\ + \frac{\frac{1}{2}(\beta^2 - 3\alpha^2)e^{2\alpha} - (\beta^2 - 3\alpha^2)\dfrac{\sin\beta}{\beta}\left[\alpha^2(\beta^2 + \alpha^2)\dfrac{\mathrm{sh}\sqrt{\beta^2 - 2\alpha^2}}{\sqrt{\beta^2 - 2\alpha^2}} + 2\alpha(\beta^2 - \alpha^2)\,\mathrm{ch}\sqrt{\beta^2 - 2\alpha^2}\right]}{2\left[(\beta^2 - 3\alpha^2)^2 + 4\alpha^2\beta^2\right]} = 0 \tag{2.9*}$$

The system of equations (2.8), (2.9) in which the velocity A is considered as a known quantity while the parameters α, β are unknown possesses the property that to each solution (α, β) there corresponds, by formula (2.7), an eigenvalue λ of the fundamental boundary problem; to each eigenvalue of the fundamental boundary problem, for given A, there corresponds at least one solution (α, β) of the system (2.8), (2.9).

We shall denote the system (2.8), (2.9) as the characteristic system. Let us consider some of its properties.

For given A let the pair of numbers (α, β) be a solution of the characteristic system. Then for the same A its solutions will also be

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*NASA Reviewer's note: In the first denominator in this equation $4\alpha - \beta^2$ should read $4\alpha^2 - \beta^2$.

$$\left.\begin{array}{l}\alpha' = \alpha \\ \beta' = -\beta\end{array}\right\}, \quad \left.\begin{array}{l}\alpha'' = \frac{1}{2}(i\beta - \sqrt{\beta^2 - 2\alpha^2}) \\ \beta'' = \frac{i}{2}(2\alpha + \sqrt{\beta^2 - 2\alpha^2} + i\beta)\end{array}\right\} \text{ etc.} \tag{2.10}$$

Such solutions, obtained from the initial solution (α, β) by a change of numeration of the roots z_i in formulas (2.5), (2.6), give the same eigenvalue as the solution (α, β). That is, in order to investigate all eigenvalues it is not necessary to consider all solutions of the characteristic system but it is sufficient to consider only a part of them, the so-called essential solutions.

The essential solutions can easily be found for A = 0. From equation (2.8) for A = 0 there is obtained either $\alpha = 0$ or $\alpha = \pm\beta$. For these cases equation (2.9) assumes respectively the forms:

$$F(0, \beta) = \frac{1}{2}(1 + \operatorname{ch}\beta\cos\beta) = 0, \quad F(\pm\beta, \beta) = \frac{1}{4}(2 + \operatorname{ch}2\beta + \cos 2\beta) = 0 \tag{2.11}$$

Consideration of the real positive roots β of the first of equations (2.11) leads to the obtaining of all essential solutions of the characteristic system for A = 0:

$$\begin{array}{lll}(\alpha_1 = 0,\ \beta_1 = 1.8751), & (\alpha_2 = 0,\ \beta_2 = 4.6941), & (\alpha_3 = 0,\ \beta_3 = 7.8548) \\ (\alpha_4 = 0,\ \beta_4 = 10.9955), & \left(\alpha_m = 0,\ \beta_m \approx \left(m - \frac{1}{2}\right)\pi\right) & (m = 5, 6, 7, \ldots)\end{array} \tag{2.12}$$

The remaining roots of the first of equations (2.11) and also the roots of the second equation need not be considered because they lead to the solutions (α, β) which are connected with the obtained solutions (2.12) precisely by formulas (2.10).

Let $(\alpha = 0, \beta = \beta_m)$ be one of the points (2.12). At the point $(0, \beta_m)$ the relations are satisfied

$$F(0, \beta_m) = 0, \qquad \frac{\partial F(0, \beta_m)}{\partial \beta} = \frac{1}{2}(\operatorname{sh}\beta_m\cos\beta_m - \operatorname{ch}\beta_m\sin\beta_m) \neq 0$$

By the theorem on implicit functions [2] (p. 354) there exists in a certain neighborhood of the point $(0, \beta_m)$ a single analytical function $\beta = \beta_m(\alpha)$ converting equation (2.9) into an identity. The function $F(\alpha, \beta)$ for real α, β is real; hence the function $\beta = \beta_m(\alpha)$ for real α assumes real values.

Let us consider the equation

$$f(A, \alpha, \beta_m(\alpha)) \equiv A - 4\alpha [\beta_m^2(\alpha) - \alpha^2] = 0 \tag{2.13}$$

which is a result of substituting $\beta = \beta_m(\alpha)$ in the right hand side of equation (2.8). At the point ($A = 0$, $\alpha = 0$) the relations $f = 0$, $\partial f/\partial\alpha = -4\beta_m^2 \neq 0$ are satisfied.

By the theorem on implicit functions there exists in a certain neighborhood of the point ($A = 0$, $\alpha = 0$) a unique analytical function $\alpha = \alpha_m(A)$ converting equation (2.13) into an identity and assuming for real A real values. The substitution $\alpha = \alpha_m(A)$ in the expression $\beta = \beta_m(\alpha)$ leads to the analytic function $\beta = \beta_m(A)$.

Thus, for sufficiently small $|A|$ there exists a unique pair of analytical functions

$$\alpha = \alpha_m(A), \qquad \beta = \beta_m(A) \tag{2.14}$$

which convert both equations of the characteristic system into identities, with $\alpha_m(0) = 0$, $\beta_m(0) = \beta_m$. The pair of functions (2.14) for varying A determines in the space of quantities α, β a unique analytical curve passing for $A = 0$ through the point $(0, \beta_m)$. To real sufficiently small A correspond real points (α,β) of the curve (2.14).

Through each point (2.12) passes its unique analytical curve (2.14), real for sufficiently small real A, constituted only of the solutions of the characteristic system. The curves (2.14) will be denoted as the branches of the characteristic system.

If for $A = 0$, besides the points (2.12), some other points (α,β) are considered — solutions of the characteristic system, then it can be found that through each such point there passes a single branch of the characteristic system; the study of these branches does not however give any new eigenvalues.

Let us consider some branch of (2.14). For a change of A in the interval $0 < A \leq A'$ let the branch exist and not intersect with any other branches of the characteristic system. Since $A' > 0$, $\alpha \neq 0$ in the neighborhood of the point ($\alpha_m' = \alpha_m(A')$, $\beta_m' = \beta_m(A')$ [see (2.8)]. In the neighborhood of this point we find from equation (2.8)

$$\beta = \sqrt{\frac{A}{4\alpha} + \alpha^2} \tag{2.15}$$

Substituting (2.15) in equation (2.9) we obtain the equation $F(A, \alpha) = 0$, the left hand side of which in the neighborhood of the point (A', α_m') is an analytical function of its arguments. At the point (A', α_m') itself the relations $F(A', \alpha_m') = 0$, $F(A', \alpha) \not\equiv 0$, $F(A, \alpha_m') \not\equiv 0$ are satisfied. The satisfying of these relations and the analytic character of $F(A, \alpha)$ are sufficient conditions for the application of the Weierstrass lemma [3] (p. 137). According to this lemma there exists in the neighborhood of the point (A', α_m') the representation

$$F(A, \alpha) = [(\alpha - \alpha_m')^r + (\alpha - \alpha_m')^{r-1} f_1(A) + \ldots + f_r(A)]\, \Phi(A, \alpha) \tag{2.16}$$

where $\underline{r}$ is a *nonnegative integer*, $f_i(A)$ is an *analytical function becoming zero* for $A = A'$ and $\Phi(A, \alpha)$ an analytical function of its arguments, not becoming zero in the neighborhood of the point (A', α_m'). From the representation (2.16) it follows that in the neighborhood of the point (A', α_m') the equation $F(A, \alpha) = 0$ is equivalent to the equation

$$(\alpha - \alpha_m')^r + (\alpha - \alpha_m')^{r-1} f_1(A) + \ldots + f_r(A) = 0 \tag{2.17}$$

For real $A \leq A'$ in the neighborhood of the point (A', α_m') there exists a single branch of (2.14), whence follows the equation $r = 1$. Now from (2.17), (2.15) we find $\alpha = \alpha_m' - f_1(A) = \alpha_m(A)$, $\beta = \beta_m(A)$. The obtained functions determine in the neighborhood of the point (α_m', β_m') a single analytical curve not only for values $A \leq A'$ but also for values $A > A'$. The meaning of the preceding considerations is that, in the first place, the branch considered cannot cease existing for $A > A'$ if it exists for $A \leq A'$, and secondly, cannot "branch out," i.e. cannot be of such character that for $A \leq A'$ in the neighborhood of the point (α_m', β_m') there exists one branch, while for $A > A'$ two or more branches.

Let $A'' > 0$ be that value of the magnitude A for which $\underline{r}$ different branches of the characteristic system have a common point (α_m'', β_m''). Applying the same considerations for the neighborhood of the point (α_m", β_m") that were applied for the neighborhood of the point (α_m', β_m') we discover with the aid of the Weierstrass lemma that the branches existing separately before intersection (for $A < A$") continue to exist separately in the same number r after intersection (for $A > A$").

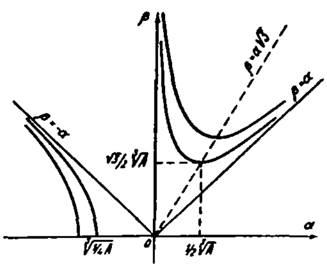

Fig. 3.

The above proven properties of "nondestructibility" of the branches of the characteristic system are easily carried over to the branches $\lambda = \lambda_m(A)$ of equation (2.4), obtained from the branches (2.14) by recomputation by formula (2.7).

We now introduce rectangular cartesian coordinates α, β in the plane and making use of the real essential branches (2.14) for small real A we come to the question of their position in the plane, α, β.

It is sufficient to conduct the investigation in the upper half of the plane α, β in view of the fact that the equations of the characteristic system are even with respect to β.

The line $\ell_1(A)$, determined by equation (2.8) for $A = 0$, consists of three straight lines $\alpha = 0$, $\alpha = \beta$, $\alpha = -\beta$ and, for $A \neq 0$, of two separate pieces each of which can be constructed point by point with the aid of relation (2.15). Fig. 3 shows the position of the curves $\ell_1(A)$ for three increasing values of A.

With increase of $A > 0$ the curves $\ell_1(A)$ are displaced upward in the first quadrant and to the left in the second quadrant.

Whatever the value of $A > 0$ the curves $\ell_1(A)$ of the first quadrant have as asymptotes the straight lines $\alpha = 0$, $\alpha = \beta$, and the curves of the second quadrant the straight line $\alpha = -\beta$.

For what follows it is useful to know how the magnitude $\lambda = (\alpha^2 + \beta^2)(\beta^2 - 3\alpha^2)$ varies along the curve $\ell_1(A)$. From (2.7), (2.8) we find that along the curve $\ell_1(A)$

$$\frac{\partial \lambda}{\partial \alpha} = -\frac{2}{\alpha}\left[(\beta^2 - 3\alpha^2)^2 + 4\alpha^2\beta^2\right]$$

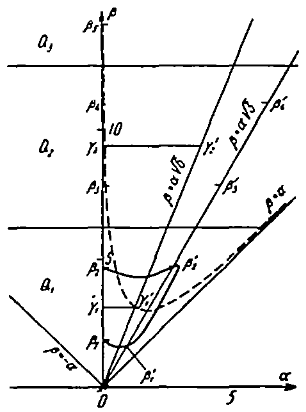

Fig. 4.

That is to say, for a motion along the curve $\ell_1(A)$ in the first quadrant from right to left (in the direction of decreasing α) the magnitude λ increases.

For $A > 0$ the points of intersection of the curve $\ell_1(A)$ with the curves ℓ_2, determined by equation (2.9), will be numbered in such manner that to a larger number m will correspond a smaller value $\alpha_m(A)$. The real eigenvalues $\lambda_m(A)$ will then increase with increase in the number m.

The curves $\ell_1(A)$ and $\ell_1(-A)$ are symmetrical with respect to the straight line $\alpha = 0$.

For convenience in studying the curves ℓ_2 of equation (2.9) we shall consider the region Q_n bounded by the segments of the lines

$$\alpha = \beta, \quad \alpha = -\beta, \quad \beta = 2n\pi \qquad (n=1, 2, \ldots)$$

The region Q_1 is triangular, the regions Q_2, Q_3 . . . . trapezoidal (Fig. 4).

It is easy to prove immediately that on the boundary of the regions Q_n we have $F(\alpha,\beta) > 0$.

In each of the regions Q_n for $A = 0$ there arise at the points (2.12) two essential branches of the characteristic system.

For any value of the velocity A in each of the regions Q_n with a sufficiently large number n the curve $\ell_1(A)$ intersects the curves ℓ_2 exactly in two real points. With the aid of (2.8), (2.9) we can show that these points, with increase in the number n, asymptotically approach the corresponding points (2.12) of the essential branches for $A = 0$, that is,

$$\alpha_m(A) \to 0, \; \beta_m(A) \to \left(m - \tfrac{1}{2}\right)\pi \quad \text{for } m \to \infty$$

Whence with the aid of formula (2.7) we conclude that the eigenvalues $\lambda_m(A)$ for sufficiently large m are real; with increase in m they asymptotically approach the eigenvalues $\lambda_m(0)$ for $A = 0$.

We denote by the symbol $Q_n(F < 0)$ that part of the region Q_n where $F(\alpha,\beta) < 0$. It is easily shown that to the regions $Q_n(F < 0)$ belong the segments $\alpha = 0$, $\beta_{2n-1} < \beta < \beta_{2n}$ $(n = 1, 2, \ldots)$, where β_m are given by the equations (2.12) (segments $\beta_1\beta_2$, $\beta_3\beta_4$, ... of the β-axis in Fig. 4), the segments $0 \leq \alpha \leq$ $\leq \beta/\sqrt{6}$, $\beta = (2n-1)\pi$ $(n = 1, 2, \ldots)$, (the segments $\gamma_1\gamma_1'$, $\gamma_2\gamma_2'$, ... in Fig. 4), and the segments $\beta = \alpha\sqrt{3}$, $\beta'_{2n-1} < \beta < \beta'_{2n}$ $(n = 1, 2, \ldots)$ where β_m' are given by the equations

$$\beta_1'=1.6020, \quad \beta_2'=4.7123, \quad \beta_3'=7.8540, \quad \beta_4=10.9956, \quad \beta_m' \approx \left(m-\frac{1}{2}\right) \; (m=5,6,\ldots) \tag{2.18}$$

(segments $\beta_1'\beta_2'$, $\beta_3'\beta_4'$, ... in Fig. 4).

The last statement for example follows from the fact that along the straight line $\beta = \alpha\sqrt{3}$ the left side of equation (2.9)

$$F(\beta/\sqrt{3}, \beta) = \frac{2}{3} e^{\beta/\sqrt{3}} \left(\cos\beta + \frac{1}{2} e^{-\beta\sqrt{3}}\right)$$

becomes zero for the values (2.18), being negative in the intervals

$$\beta_{2n-1} < \beta < \beta_{2n}'$$

Let us consider the curve $\ell_1(A)$ passing through the point γ_1' of the region $Q_1(F<0)$ with coördinates $(\pi/\sqrt{6}, \pi)$ (the dotted curve in Fig. 4). The corresponding value of A is equal to $10\pi^3/3\sqrt{6}$.

For this value of A, and also for any value of A from the interval $0 \leq A \leq$ $\leq 10\pi^3/3\sqrt{6}$ the curve $\ell_1(A)$ in each of the regions Q_n intersects the segment $\gamma_n\gamma_n'$ of the region $Q_n(F<0)$. From this and from the "indestructibility" property it follows that for any value of A from the interval under consideration the curve $\ell_1(A)$ in each of the regions Q_n has exactly two different real intersections with the lines ℓ_2. In the regions Q_2, Q_3, ... these intersections are located to the left of the straight line $\beta = \alpha\sqrt{3}$ and hence the eigenvalues $\lambda_m(A)$ for $m > 2$ are all positive.

In the region Q_1 the curve $\ell_1(A)$ for the value $A = 10\pi^3/3\sqrt{6}$ intersects the segment β_1', β_2' of the region $Q_1(F < 0)$. From this we conclude that any points of the segments $\beta_1\beta_2$, $\gamma_1\gamma_1'$, $\beta_1'\beta_2'$, for which $F(\alpha,\beta) < 0$ can be connected with each other by a continuous curve not going out beyond the limits of the region $Q_1(F < 0)$. That is to say, the part of the region $Q_1(F < 0)$ adjoining the segments $\beta_1\beta_2$, $\gamma_1\gamma_1'$, β_1', β_2' in the first quadrant has qualitatively the appearance shown in Fig. 4.

For those values of A for which the curve $\ell_1(A)$ intersects the internal points of the segment $\beta_1'\beta_2'$ of the straight line $\beta = \alpha\sqrt{3}$ the eigenvalues $\lambda_2(A)$, $\lambda_3(A)$, ... are different, real and positive, and the first eigenvalue $\lambda_1(A)$ is negative (the first point of intersection of the curves $\ell_1(A)$ and ℓ_2 is located to the right of the straight line $\beta = \alpha\sqrt{3}$).

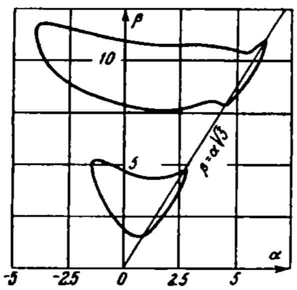

Fig. 5.

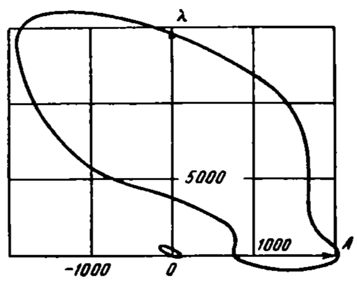

Fig. 6.

Through the points β_1', β_2' there pass the curves $\ell_1(A)$ with the values $A = 6.33$ and $A = 161$. For any A of the interval $0 \leq A < 6.33$ all eigenvalues of the fundamental boundary problem are different, real and positive. For $A = 6.33$ all eigenvalues are different, real and, except for λ_1, positive, while the first eigenvalue is equal to zero. For any A of the interval $6.33 < A < 161$ all eigenvalues are different, real and, except for λ_1, positive, while the first eigenvalue is negative.

Let A^* be the upper bound of those values of A for which the curve $\ell_1(A)$ has real points in common with the region $Q_1(F < 0)$. In the neighborhood of the value of A^* for $A < A^*$ the curve $\ell_1(A)$ in the region Q_1 has two different real intersections with the curve ℓ_2, while for $A > A^*$ it has not a single real intersection. On account of the indestructibility in the neighborhood of A^* for $A > A^*$ the curve $\ell_1(A)$ has two different complex intersections (the points of the first and second essential branches of the characteristic system become complex). The corresponding values of $\lambda_1(A)$ and $\lambda_2(A)$ become conjugate complex numbers.

The results of the numerical computations entirely confirm the preceding conclusions. With the aid of the representations $\alpha = \alpha + C_1F(\alpha,\beta)$ or $\beta = \beta + C_2F(\alpha,\beta)$, where C_1, C_2 are constants different from zero, individual points (α,β) of the curves ℓ_2 situated in the regions Q_1, Q_2, were found by the method of successive approximations. The curves ℓ_2 are shown in Fig. 5. For those points (α,β) however by which the curves of Fig. 5 were constructed the corresponding points (A, λ) were found with the aid of formulas (2.7), (2.8) and the curves of Fig. 6 constructed which give an idea of the change of the first four eigenvalues of the fundamental boundary problem with change in the velocity A. Fig. 7 shows to a larger scale part of Fig. 6 near the origin.

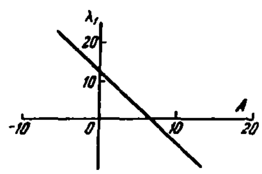

Fig. 7.

As supplementary to Figs. 6, 7 we present below the data on the behavior of the first four eigenvalues for various values of the velocity A:

$0 \leqslant A < 6.33$,	$\lambda_1 > 0$;	$745 \leqslant A < 746$,	$\lambda_1 > 0$
$A = 6.33$,	$\lambda_1 = 0$;	$A = 746$,	$\lambda_1 = 0$
$6.33 < A \leqslant 162$,	$\lambda_1 < 0$;	$746 < A \leqslant 2047$,	$\lambda_1 < 0$
$162 < A < 745$,	λ_1†	$2047 < A < *$,	λ_1†
$0 \leqslant A < 161$,	$\lambda_2 > 0$;	$777 < A < 1672$,	λ_2†
$A = 161$,	$\lambda_2 = 0$;	$1672 \leqslant A < 2046.7$,	$\lambda_2 > 0$
$161 < A \leqslant 162$,	$\lambda_2 < 0$;	$A = 2046.7$,	$\lambda_2 = 0$
$162 < A < 745$,	λ_2†	$2046.7 < A \leqslant 2047$,	$\lambda_2 < 0$
$745 \leqslant A \leqslant 777$,	$\lambda_2 > 0$;	$2047 < A < *$,	λ_2†
$0 \leqslant A \leqslant 777$,	$\lambda_3 > 0$;	$1672 \leqslant A \leqslant 1688$,	$\lambda_3 > 0$
$777 < A < 1672$,	λ_3†	$1688 < A < *$,	λ_3†
$0 \leqslant A \leqslant 1688$,	$\lambda_4 > 0$;	$1688 < A < *$,	λ_4†

(2.19)

The asterisks indicate values not determined by us. Multiple eigenvalues were noted in the following cases:

$A = 162$,	$\lambda_1 = \lambda_2 = -26$;	$A = 1672$,	$\lambda_2 = \lambda_3 = 3392$
$A = 745$,	$\lambda_1 = \lambda_2 = 41$;	$A = 1688$,	$\lambda_3 = \lambda_4 = 5413$
$A = 777$,	$\lambda_2 = \lambda_3 = 902$;	$A = 2047$,	$\lambda_1 = \lambda_2 = -43$

† No real values for (λ)'s found.

3. Character of the natural motions for different velocities of motion of the plate. Critical velocity. For zero value of the velocity A all eigenvalues $\lambda_m(0)$ of the fundamental boundary problem are different, real and positive, and are located in the complex plane within the stability parabola; moreover

$$\lambda_m(0) \to (m - \tfrac{1}{2})^4 \pi^4 \quad \text{for} \quad m \to \infty$$

The complex frequencies, with the possible exception of several of the first frequencies, are given by the formulas

$$\omega_m = p_m \pm i q_m, \qquad p_m = -\frac{B}{2h\rho}, \qquad q_m = \frac{1}{2h\rho}\sqrt{\frac{2h\rho D\lambda_m}{a^4} - B^2}$$

All natural motions have an amplitude decreasing with time; if they have an oscillatory character they are similar to standing waves.

Whatever the velocity $A \neq 0$, all natural motions corresponding to $\lambda_m(A)$ with sufficiently large numbers m, have the same character that they had for $A = 0$. This assertion, which follows from the asymptotic approach of $\lambda_m(A)$ to $\lambda_m(0)$ as $m \to \infty$, means, in other words, that the high frequency vibrations of the plate are less distorted by the oncoming gas flow the larger their frequency.

For any value of the velocity A in the interval $0 \leq A < 6.33$ the degree of instability of the undisturbed motion is equal to zero. All natural motions have an amplitude which decreases with time.

For any value of the velocity A in the interval $6.33 < A < 161$ the degree of instability of the undisturbed motion is equal to unity. One of the natural motions, corresponding to the first eigenvalue, has an amplitude increasing with time; the motion has the character of an aperiodic deviation from the undisturbed state and is termed the divergent motion or simply the divergence. The remaining natural motions have an amplitude which decreases with time. The interval $6.33 < A < 161$ belongs to the instability range of the undisturbed motion.

The value of the velocity $A = 6.33$ on passing through which the degree of instability of the undisturbed motion changes from zero to one we shall denote as the critical. For $A = 6.33$ all eigenvalues $\lambda_m(6.33)$ are different, real and except the first, positive, the first eigenvalue being equal to zero. One of the natural motions, corresponding to this zero λ, is the motion of the plate with constant deflection $w(x)$ independent of the time. The remaining natural motions have an amplitude decreasing with time.

For any value of the velocity A in the interval $161 < A < 162$ all eigenvalues $\lambda_m(A)$ of the fundamental boundary problem are different, real and, except $\lambda_1(A)$ and $\lambda_2(A)$, positive, the first two values being negative; the degree of instability of the undisturbed motion is equal to two. Two independent natural motions have the character of an aperiodic deviation from the undisturbed state, that is, the character of divergence. The remaining natural motions have amplitudes decreasing with time. The interval $161 < A < 162$ belongs to the instability range of the undisturbed motion.

The value of the velocity $A = 161$ on passing through which the degree of instability of the undisturbed motion increases from one to two we shall likewise denote as critical. For $A = 161$ all eigenvalues $\lambda_m(161)$ of the fundamental boundary problem are different, real and, except $\lambda_1(161)$ and $\lambda_2(161)$, positive; the first eigenvalue is negative while the second is equal to zero. The value $A = 161$ belongs to the instability range of the undisturbed motion.

For any value A of the interval $162 \leq A \leq 745$ the eigenvalues $\lambda_3(A)$, $\lambda_4(A)$, ... are different, real, positive and located in the complex plane within the stability parabola; the degree of instability is determined by the behavior of $\lambda_1(A)$, $\lambda_2(A)$.

For $A = 162$ the two first eigenvalues of the fundamental boundary problem coincide, $\lambda_1(162) = \lambda_2(162) = -26$. In the neighborhood of $A = 162$ for $A > 162$ the eigenvalues $\lambda_1(A)$ and $\lambda_2(A)$ are complex conjugate numbers, situated in the complex plane outside the limits of the stability parabola. In this neighborhood, belonging to the instability range of the undisturbed motion, the degree of instability, as in the interval $161 < A < 162$, is equal to two; however, the natural motions with increasing amplitude, which correspond to the eigenvalues $\lambda_1(A)$ and $\lambda_2(A)$, have a different character, which now resembles waves of a flag in windy weather. Such wave motions with increasing amplitude are denoted as flutter motions or simply flutter.

Thus, as the velocity A passes through the value $A = 162$ the divergence of the plate changes to flutter.

For A = 745 the two first eigenvalues of the fundamental boundary problem again coincide, $\lambda_1(745) = \lambda_2(745) = 41$. In the neighborhood of A = 745 for A < 745 the eigenvalues $\lambda_1(A)$ and $\lambda_2(A)$ are complex conjugate numbers situated in the complex plane within the stability parabola. In this neighborhood the degree of instability of the undisturbed motion is equal to zero.

For a certain critical value A in the interval 162 < A < 745 the degree of instability of the undisturbed motion changes from two to zero; the flutter natural motions are changed to motions with decreasing amplitude.

The degree of instability of the undisturbed motion remains equal to zero also in the interval 745 < A < 746, since for any value of A in this interval all eigenvalues of the fundamental boundary problem are different, real and positive, being situated in the complex plane within the stability parabola.

The investigation of the natural motions for the other regions (2.19) of variation of the velocity A offers no difficulties and we omit it.

In the problem considered there exist infinitely many critical values of the velocity A in passing through which the degree of instability of the undisturbed motion changes. For example, to each point (2.18) of the straight line $\beta = \alpha\sqrt{3}$ corresponds, according to (2.8), its critical value of the reduced velocity $A_m = 8\beta_m'^3/3\sqrt{3}$ (m = 1, 2, ...), where any of the intervals $A_{2m-1} < A < A_{2m}$, m = 1, 2, ..., as is known, belongs to the instability range of the undisturbed motion.

The first, smallest critical value of the reduced velocity $A_1 = 6.33$, is of most interest. Assuming that the reduced velocity A is connected with the velocity c of the undisturbed motion of the plane by formula (1.4), we obtain the smallest critical velocity:

$$c_k = 0.264 \frac{c_0}{p_0 \varkappa} \frac{E}{1-\nu^2} \left(\frac{2h}{a}\right)^3 \qquad (3.1)$$

As an example let us consider a steel plate with the elasticity constants $E = 2.1 \cdot 10^{10}\,\mathrm{kg/m^2}$, $\nu = 0.3$, moving in a gas whose state is characterized by the constants $\kappa = 1.4$, $p_0 = 103 \cdot 10^2\,\mathrm{kg/m^2}$, $c_0 = 340$ m/sec (air at sea level at temperature 15°). The computation by formula (3.1) gives

$$c_k = 143 \cdot 10^6 \left(\frac{2h}{a}\right)^3 \ \mathrm{m/sec} \qquad (3.2)$$

For different values of the magnitude $a/2h$ there were computed by formula (3.2) the values c_k. The results are shown in Fig. 8.

In Figs. 5, 6, together with the curves in the first quadrant, there are shown curves in the second quadrant that correspond to negative values of the magnitude A, that is, to undisturbed motion of the plate in a direction opposite to that which was initially assumed. For such undisturbed motion of the plate in the opposite direction in the interval $0 \leq -A < < 135$ we were not successful in finding any critical values of the velocity. For $A = -135$ the first two eigenvalues of the fundamental boundary problem coincide: $\lambda_1(-135) = \lambda_2(-135) = 649$. With further increase in the modulus of the velocity A the eigenvalues $\lambda_1(A)$ and $\lambda_2(A)$ become complex conjugate numbers. It is possible that at a certain velocity $-A > 135$ they issue beyond the limits of the stability parabola and flutter then occurs.

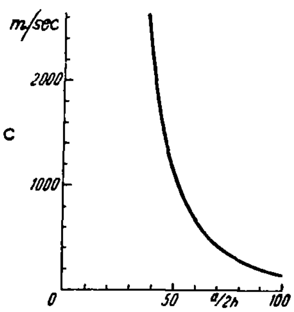

Fig. 8.

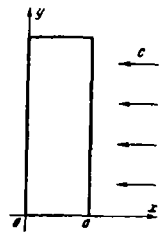

Fig. 9.

In computing the critical velocity c_k we assumed that the reduced velocity A was connected with the characteristic of the plate, the gas, and the velocity of the undisturbed motion $\underline{c}$ by the formula (1.4). The obtained results may be applied to cases of a more general dependence of the reduced velocity A on its arguments, assuming for example for A the expression

$$A = \frac{2a^3 p_0 \varkappa c}{D c_0} \frac{c / c_0}{\sqrt{(c / c_0)^2 - 1}} \tag{3.3}$$

We have considered a plate of infinite span. The results evidently can refer also to rectangular plates for which the dimension along the y-axis is several times greater than the dimension along the x-axis. Such cantilevers can easily be mounted in an artificially produced flow directed from the free to the clamped edge of the plate (Fig. 9). Having established a constant supersonic velocity $\underline{c}$ and gradually increasing the dimension $\underline{a}$ of the plate up to rupture, it is then possible to apply (1.4), (3.3), or some other expression for A, in order to compare the theoretical with the experimental results.

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